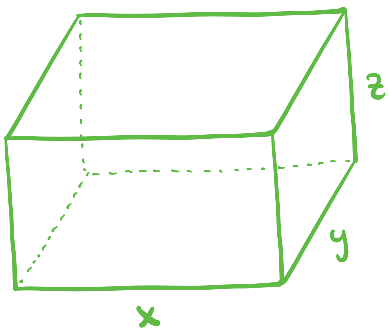
1. (10 points) A rectangular box without a lid has volume $4\,\text{m}^3$. Find the minimum possible surface area. No points will be awarded for guessing.




Width $x$, length $y$, height $z$

$$\Rightarrow \begin{cases} \text{volume} = xyz = 4. \\ \text{surface area} = xy + 2yz + 2zx. \end{cases}$$

Set $f(x,y,z) = xy + 2yz + 2zx$ and $g(x,y,z) = xyz - 4$.

Solve $\nabla f = \lambda \nabla g$ and $g = 0$

$\Rightarrow (y+2z,\ x+2z,\ 2x+2y) = \lambda(yz, zx, xy)$ and $xyz - 4 = 0$.

$$\Rightarrow \begin{cases} y+2z = \lambda yz & \rightsquigarrow & xy + 2xz = \lambda xyz \\ x+2z = \lambda zx & \rightsquigarrow & xy + 2yz = \lambda xyz \\ 2x+2y = \lambda xy & \rightsquigarrow & 2xz + 2yz = \lambda xyz \end{cases}$$

$\Rightarrow xy + 2xz = xy + 2yz = 2xz + 2yz$

$\Rightarrow xy = 2yz = 2zx \Rightarrow \dfrac{xyz}{xy} = \dfrac{xyz}{2yz} = \dfrac{xyz}{2zx} \Rightarrow z = \dfrac{x}{2} = \dfrac{y}{2}$

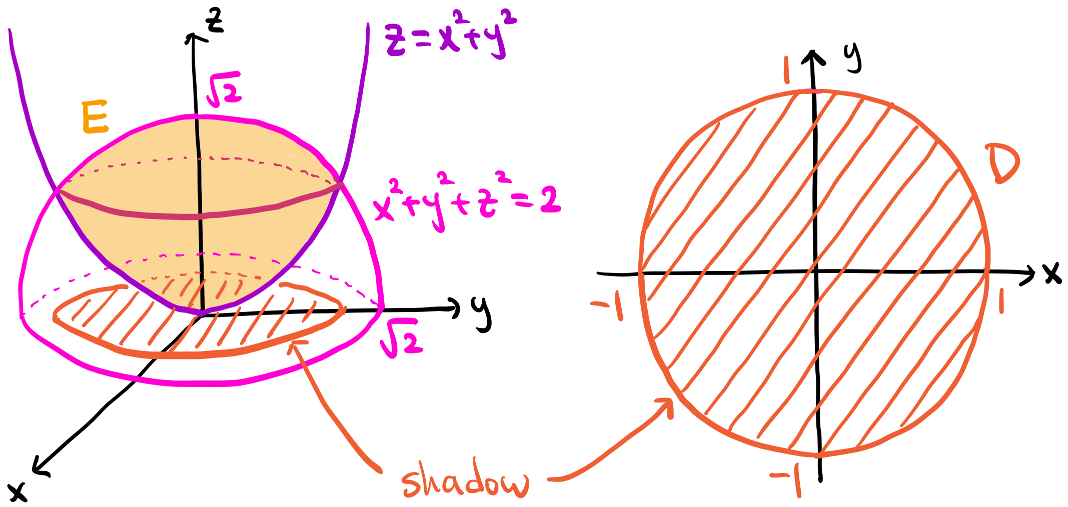
$xyz = 4 \Rightarrow x = 2,\ y = 2,\ z = 1.$

The minimum surface area is $f(2,2,1) = \boxed{12\ (\text{m}^2)}$

<u>Note</u> Alternatively, you can remove the constraint $xyz = 4$ by writing $z = \dfrac{4}{xy}$ and find the minimum of

$xy + 2yz + 2zx = xy + 2y \cdot \dfrac{4}{xy} + 2 \cdot \dfrac{4}{xy} \cdot x$ on the open domain given by $x > 0$ and $y > 0$.

2. (10 points) Find the volume of the region bounded above by the spherical surface $x^2+y^2+z^2=2$ and below by the paraboloid surface $z=x^2+y^2$.




In cylindrical coordinates:

$x^2+y^2+z^2=2 \rightsquigarrow r^2+z^2=2 \rightsquigarrow z=\sqrt{2-r^2}$ $z\geq 0$

$z=x^2+y^2 \rightsquigarrow z=r^2$

Intersection: $z=\sqrt{2-r^2}$ and $z=r^2 \Rightarrow z=\sqrt{2-z}$

$\Rightarrow z^2=2-z \Rightarrow z=1, \cancel{-2} \Rightarrow r=1.$ $z\geq 0$

The shadow on the $xy$-plane: $0\leq\theta\leq 2\pi,\ 0\leq r\leq 1.$

For each point on the shadow: $r^2\leq z\leq\sqrt{2-r^2}.$

$\Rightarrow 0\leq\theta\leq 2\pi,\ 0\leq r\leq 1,\ r^2\leq z\leq\sqrt{2-r^2}.$

Jacobian

$$\text{Volume} = \iiint_E 1\,dV = \int_0^{2\pi}\int_0^1\int_0^{\sqrt{2-r^2}} 1\cdot r\,dz\,dr\,d\theta$$

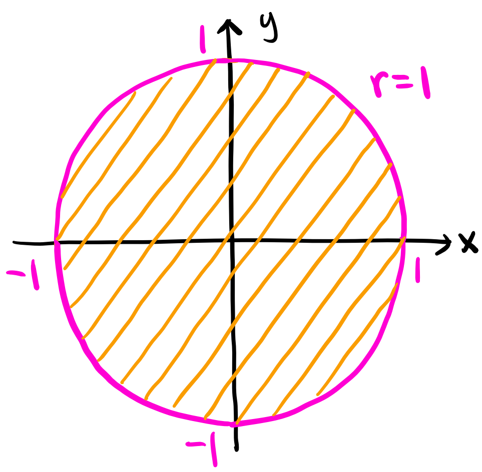
$$= \int_0^{2\pi}\int_0^1 (\sqrt{2-r^2}-r^2)\,r\,dr\,d\theta \underset{u=2-r^2}{=} \int_0^{2\pi}\int_2^1 (u^{1/2}-2+u)\left(-\tfrac{1}{2}\right)du\,d\theta$$

$$= \int_0^{2\pi} -\frac{u^{3/2}}{3}+u-\frac{u^2}{4}\Big|_{u=2}^{u=1} d\theta = \int_0^{2\pi}\frac{8\sqrt{2}-7}{12}\,d\theta = \boxed{\frac{\pi(8\sqrt{2}-7)}{6}}$$

3. Find the area of the loops, which are given by

a) (3 points) $r = 1$ and $0 \leq \theta < 2\pi$,

The loop is the circle of radius 1 and center $(0,0)$.




$\Rightarrow$ The area is $\pi \cdot 1^2 = \boxed{\pi}$

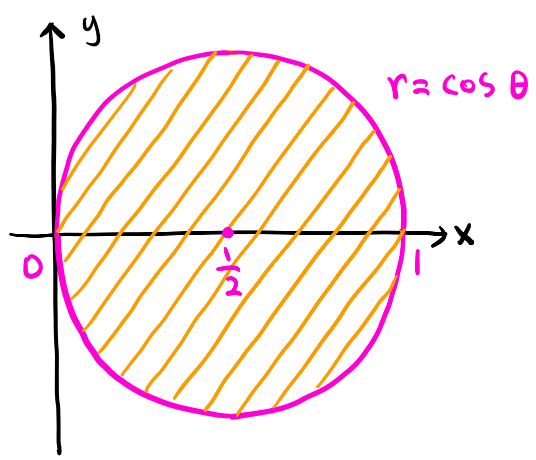
b) (7 points) $r = \cos\theta$ and $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$,

in polar coordinates. You may use the formula $\cos^2\theta = \frac{1+\cos 2\theta}{2}$.

$$r = \cos\theta \rightsquigarrow r^2 = r\cos\theta \rightsquigarrow x^2 + y^2 = x$$

$$\rightsquigarrow x^2 - x + \frac{1}{4} + y^2 = \frac{1}{4} \rightsquigarrow \left(x - \frac{1}{2}\right)^2 + y^2 = \frac{1}{4}.$$

The loop is the circle of radius $\frac{1}{2}$ and center $\left(\frac{1}{2}, 0\right)$



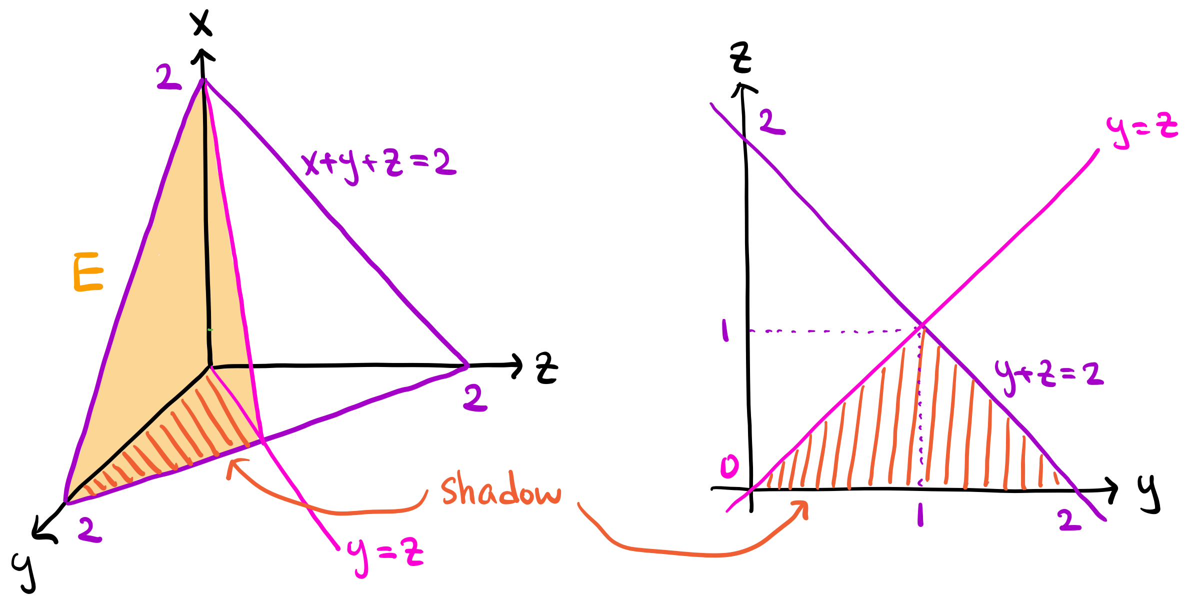

$\Rightarrow$ The area is $\pi \cdot \left(\frac{1}{2}\right)^2 = \boxed{\frac{\pi}{4}}$

Note In each part, you can also compute the area using a double integral.

4. (10 points) Find the volume of the tetrahedron whose faces are given by the planes $x = 0$, $z = 0$, $y = z$, $x + y + z = 2$. Express your answer as a simple fraction.

$x+y+z=2$ is a plane

$x$-intercept $=2$, $y$-intercept $=2$, $z$-intercept $=2$.




The base on the $yz$-plane has area $\frac{1}{2}\cdot 2\cdot 1 = 1$.

The height of the tetrahedron is 2

$\Rightarrow$ Volume $= \frac{1}{3}\cdot 1\cdot 2 = \boxed{\frac{2}{3}}$

Note You can also compute the volume by a triple integral.

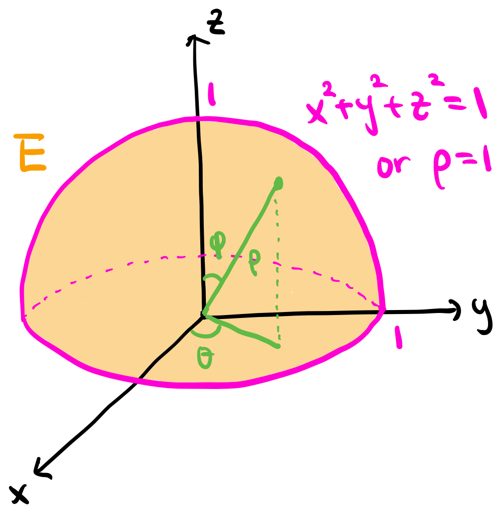
The base on the $yz$-plane : $0 \le z \le 1$, $z \le y \le 2-z$.

For each point on the base : $0 \le x \le 2-y-z$

$$\Rightarrow \text{Volume} = \int_0^1 \int_z^{2-z} \int_0^{2-y-z} 1 \, dx\, dy\, dz$$

5. Each of the three problems below is about the center of mass of the solid hemisphere $0 \leq x^2 + y^2 + z^2 \leq 1$ and $z \geq 0$.

a) (4 points) Assume the density is $\rho(x, y, z) = r$, where $r$ is the distance from the origin. Find the $z$-coordinate of the center of mass. Express your answer as a simple fraction.




In spherical coordinates:

$x^2+y^2+z^2=1 \rightsquigarrow \rho^2=1 \rightsquigarrow \rho=1$

$\varphi$ is maximized on the $xy$-plane

$\Rightarrow$ The solid $E$ is given by

$0 \leq \theta \leq 2\pi,\ 0 \leq \varphi \leq \frac{\pi}{2},\ 0 \leq \rho \leq 1$

$$m = \iiint_E \rho(x,y,z)\,dV = \iiint_E \sqrt{x^2+y^2+z^2}\,dV$$

Jacobian

$$= \int_0^{2\pi}\int_0^{\pi/2}\int_0^1 \rho \cdot \underline{\rho^2 \sin\varphi}\,d\rho\,d\varphi\,d\theta = \int_0^{2\pi}\int_0^{\pi/2} \frac{\rho^4}{4}\sin\varphi \Big|_{\rho=0}^{\rho=1} d\varphi\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/2} \frac{1}{4}\sin\varphi\,d\varphi\,d\theta = \int_0^{2\pi} -\frac{1}{4}\cos\varphi \Big|_{\varphi=0}^{\varphi=\pi/2} d\theta = \int_0^{2\pi}\frac{1}{4}\,d\theta = \frac{\pi}{2}$$

$$\bar{z} = \frac{1}{m}\iiint_E z\,\rho(x,y,z)\,dV = \frac{2}{\pi}\iiint_E z\sqrt{x^2+y^2+z^2}\,dV$$

Jacobian

$$= \frac{2}{\pi}\int_0^{2\pi}\int_0^{\pi/2}\int_0^1 \rho^2\cos\varphi \cdot \underline{\rho^2\sin\varphi}\,d\rho\,d\varphi\,d\theta$$

$$= \frac{2}{\pi}\int_0^{2\pi}\int_0^{\pi/2} \frac{\rho^5}{5}\cos\varphi\sin\varphi \Big|_{\rho=0}^{\rho=1} d\varphi\,d\theta$$

$$= \frac{2}{\pi}\int_0^{2\pi}\int_0^{\pi/2} \frac{1}{5}\cos\varphi\sin\varphi\,d\varphi\,d\theta \underset{u=\sin\theta}{=} \frac{2}{\pi}\int_0^{2\pi}\int_0^1 \frac{1}{5}u\,du\,d\theta$$

$$= \frac{2}{\pi}\int_0^{2\pi} \frac{1}{10}u^2 \Big|_{u=0}^{u=1} d\theta = \frac{2}{\pi}\int_0^{2\pi}\frac{1}{10}\,d\theta = \boxed{\frac{2}{5}}$$

b) (Hint, 0 points). If the density is $\rho(x, y, z) = 1$, the $z$-coordinate of the center of mass is $\bar{z} = \frac{3}{8}$, which you may assume. Your answer to part (a) must be greater than 3/8.

$$m = \iiint_E \rho(x,y,z)\,dV = \iiint_E 1\,dV = \text{vol}(E) = \frac{1}{2}\cdot \underbrace{\frac{4\pi}{3}\cdot 1^3}_{\text{volume of sphere}} = \frac{2\pi}{3}.$$

$$\bar{z} = \frac{1}{m}\iiint_E z\rho(x,y,z)\,dV = \frac{3}{2\pi}\iiint_E z\,dV = \frac{3}{8}.$$

c) (6 points) Suppose the density is $\rho(x, y, z) = 2r + 3$. Find the $z$-coordinate of the center of mass. It can be helpful to use your answer to (a) and the information in (b).

$$m = \iiint_E \rho(x,y,z)\,dV = \iiint_E 2\sqrt{x^2+y^2+z^2} + 3\,dV$$

$$= \underbrace{2\iiint_E \sqrt{x^2+y^2+z^2}\,dV}_{\text{mass in (a)}} + \underbrace{3\iiint_E 1\,dV}_{\text{mass in (b)}}$$

$$= 2\cdot\frac{\pi}{2} + 3\cdot\frac{2\pi}{3} = 3\pi.$$

$$\bar{z} = \frac{1}{m}\iiint_E z\rho(x,y,z)\,dV = \frac{1}{3\pi}\iiint_E z\left(2\sqrt{x^2+y^2+z^2}+3\right)dV$$

$$= \frac{1}{3\pi}\left(\underbrace{2\iiint_E z\sqrt{x^2+y^2+z^2}\,dV}_{m\bar{z}\text{ in (a)}} + \underbrace{3\iiint_E z\,dV}_{m\bar{z}\text{ in (b)}}\right)$$

$$= \frac{1}{3\pi}\left(2\cdot\frac{\pi}{2}\cdot\frac{2}{5} + 3\cdot\frac{2\pi}{3}\cdot\frac{3}{8}\right) = \boxed{\frac{23}{60}}$$

__Note__ You can directly compute the integrals in (c).

6. The two problems below are about surface areas.

a) (3 points) Find the surface area of $z = x^2 + y^2$ above the region $D$: $0 \le x^2 + y^2 \le 1$.

The surface is the graph of $f(x,y) = x^2+y^2$.

In polar coordinates, D is given by $0 \le \theta \le 2\pi$, $0 \le r \le 1$

$$\text{Area} = \iint_D \sqrt{1+f_x^2+f_y^2}\, dA = \iint_D \sqrt{1+4x^2+4y^2}\, dA$$

Jacobian

$$= \int_0^{2\pi}\int_0^1 \sqrt{1+4r^2}\cdot r\, dr\, d\theta = \int_0^{2\pi}\int_1^5 u^{\frac{1}{2}}\cdot \frac{1}{8}\, du\, d\theta$$

$u = 1+4r^2$

$$= \int_0^{2\pi} \frac{1}{12} u^{3/2}\Big|_{u=1}^{u=5} d\theta = \int_0^{2\pi} \frac{1}{12}(5^{3/2}-1)\, d\theta = \boxed{\frac{\pi}{6}(5^{3/2}-1)}$$

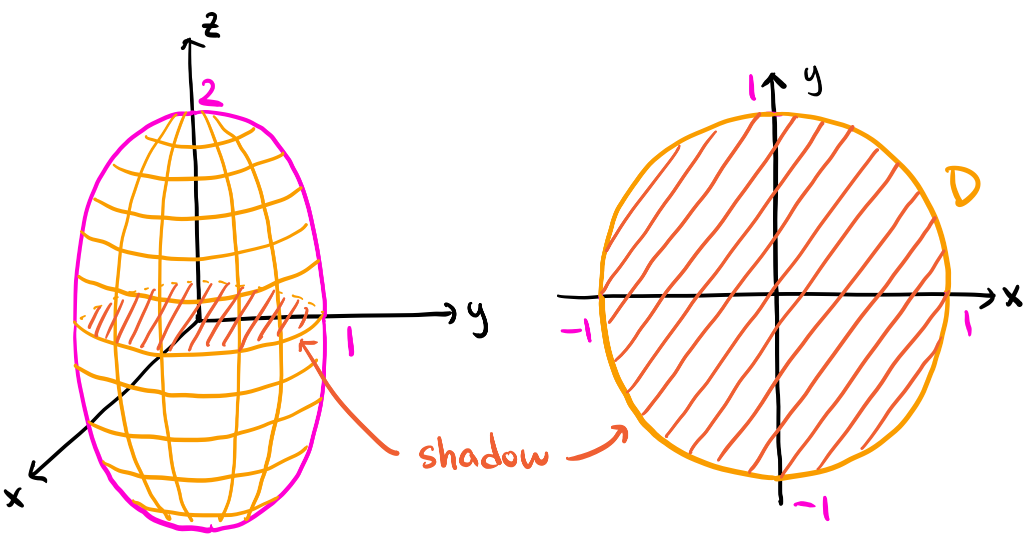
b) (7 points) The area of the ellipsoidal surface

$$x^2 + y^2 + \frac{z^2}{4} = 1$$

is equal to

$$C\int_0^1 \frac{r(1+3r^2)^{1/2}}{(1-r^2)^{1/2}}\, dr$$

for a constant $C$. Find the constant $C$.



The shadow D on the xy-plane is given by $x^2+y^2 \le 1$.

In polar coordinates, D is given by $0 \le \theta \le 2\pi$, $0 \le r \le 1$

The area of the ellipsoid is twice the area of the upper half.

$$\Rightarrow \text{Area} = 2\iint_D \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\, dA.$$

The ellipsoid is a level curve of $f(x,y,z) = x^2+y^2+\frac{z^2}{4}$.

$$\Rightarrow \begin{cases} \dfrac{\partial z}{\partial x} = -\dfrac{f_x}{f_z} = -\dfrac{2x}{z/2} = -\dfrac{4x}{z} \\[2ex] \dfrac{\partial z}{\partial y} = -\dfrac{f_y}{f_z} = -\dfrac{2y}{z/2} = -\dfrac{4y}{z} \end{cases}$$

$$\text{Area} = 2\iint_D \sqrt{1+\frac{16x^2}{z^2}+\frac{16y^2}{z^2}}\, dA = 2\iint_D \sqrt{\frac{16x^2+16y^2+z^2}{z^2}}\, dA$$

$$x^2+y^2+\frac{z^2}{4} = 1 \rightsquigarrow z^2 = 4-4x^2-4y^2$$

$$\Rightarrow \text{Area} = 2\iint_D \sqrt{\frac{16x^2+16y^2+4-4x^2-4y^2}{4-4x^2-4y^2}}\, dA$$

Jacobian

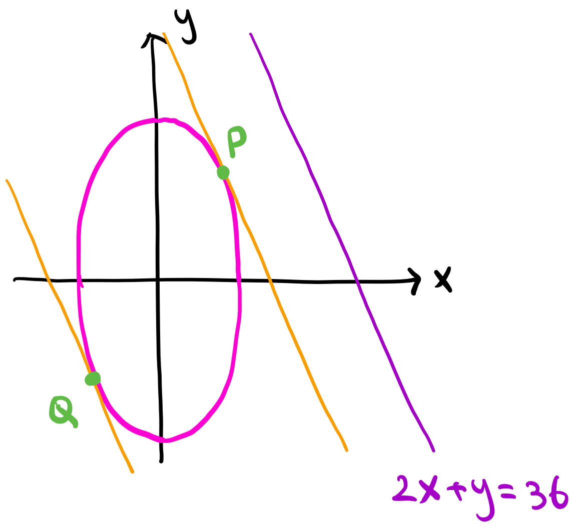
$$= 2\int_0^{2\pi}\int_0^1 \sqrt{\frac{16r^2+4-4r^2}{4-4r^2}} \cdot r\, dr\, d\theta$$

$$= 2\int_0^1\int_0^{2\pi} \sqrt{\frac{1+3r^2}{1-r^2}}\, d\theta\, dr = 4\pi\int_0^1 \sqrt{\frac{1+3r^2}{1-r^2}}\, dr$$

$$\Rightarrow C = \boxed{4\pi.}$$

7. (10 points) Find the points on the ellipse $\frac{x^2}{4} + \frac{y^2}{16} = 2$ that are closest to and furthest from the straight line $2x + y = 36$ in the $x$-$y$ plane.

very tricky!




P: the point on the ellipse closest to the line $2x+y=36$.

Q: the point on the ellipse furthest from the line $2x+y=36$

The ellipse is a level curve of $f(x,y) = \frac{x^2}{4} + \frac{y^2}{16}$.

At P and Q, the tangent line to the ellipse must be parallel to the line $2x+y=36$.

$\Rightarrow$ $\nabla f$ must be perpendicular to the line $2x+y=36$.

$2x+y=36 \rightsquigarrow y = 36-2x$

$\Rightarrow$ The direction vector of the line $2x+y=36$ is $\vec{v} = (1,-2)$

$$\nabla f \cdot \vec{v} = \vec{0} \Rightarrow \left(\frac{x}{2}, \frac{y}{8}\right)\cdot(1,-2) = 0 \Rightarrow \frac{x}{2} - \frac{y}{4} = 0 \Rightarrow y = 2x$$

$$\frac{x^2}{4} + \frac{y^2}{16} = 2 \Rightarrow \frac{x^2}{4} + \frac{(2x)^2}{16} = 2 \Rightarrow \frac{x^2}{2} = 2 \Rightarrow x = \pm 2.$$

$\Rightarrow (x,y) = (2,4), (-2,-4)$.

$\Rightarrow$ $\boxed{P = (2,4),\ Q = (-2,-4)}$